広島大学学位請求論文

Multifractal rigidity for piecewise linear Markov maps
(区分的線型写像による力学系のマルチフラクタルの剛性)

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主論文

Stochastics and Dynamics

Multifractal Rigidity for Piecewise Linear Markov Maps

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We give an answer to the multifractal rigidity problem presented by Barreira, Pesin and Schmeling for the dimension spectra of Markov measures on the repellers of piecewise linear Markov maps with two branches. Thermodynamic formalism provides us with a one-parameter family of measures. Zero-temperature limit measures of this family and the concept of nondegeneracy of spectra play important roles.

Keywords: Multifractal analysis; Multifractal rigidity; Zero-temperature limit measures

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1. Introduction

Let X be a compact metric space and $f : X \to X$ a continuous mapping. Once an invariant local quantity g and a positive set function G are given, we can define the function

$$\alpha \mapsto G(\{x \in X \mid g(x) = \alpha\}),$$

as a quantification of the complexity of the dynamical system (X, f). This function is called the *multifractal spectrum* with respect to g and G, and it provides us with a practical tool for the numerical study of the system.

We can take Birkhoff averages, Lyapunov exponents, pointwise dimensions, or local entropies as g and the Hausdorff dimension or the topological entropy as G, for example. In this paper, we consider the dimension spectra for invariant measures, which are the multifractal spectra with respect to pointwise dimensions and the Hausdorff dimension. Dimension spectra for conformal hyperbolic dynamical systems are well understood via thermodynamic formalism. In particular, [4, 10] established the multifractal formalism of dimension spectra via thermodynamical approach for the repellers of one-dimensional Markov maps. Refer to [1, 9] for related topics.

Let μ an f-invariant Borel probability measure on X. Fix $x \in X$. We write

$$d_\mu(x) = \lim_{r\to 0} \frac{\log \mu(B(x;r))}{\log r}$$

whenever the limit exists, where $B(x;r)$ denotes the closed ball of radius r center x. $d_\mu(x)$ is called the *pointwise dimension* or *local dimension* of μ at x. We set

$$X_\alpha = X_\alpha^{(\mu)} = \{x \in X \mid d_\mu(x) = \alpha\} \tag{1.1}$$

and

$$\alpha_{\min}^{(\mu)} = \inf\{\alpha \mid X_\alpha \neq \emptyset\}, \quad \alpha_{\max}^{(\mu)} = \sup\{\alpha \mid X_\alpha \neq \emptyset\}.$$

We define the function $\mathcal{D}^{(\mu)} : [\alpha_{\min}^{(\mu)}, \alpha_{\min}^{(\mu)}] \to \mathbb{R}$ by

$$\mathcal{D}^{(\mu)}(\alpha) = \dim_H X_\alpha,$$

where $\dim_H Z$ denotes the Hausdorff dimension of $Z \subset X$. We call $\mathcal{D}^{(\mu)}$ the *dimension spectrum* of μ. The interval $[\alpha_{\min}^{(\mu)}, \alpha_{\max}^{(\mu)}]$ is called the *domain* of the spectrum. $\mathcal{D}^{(\mu)}$ has much information about (X, f, μ) and we say that a *multifractal rigidity* holds if the spectrum restores the dynamical system.

In this paper, we consider the multifractal rigidity problem when (X, f) is the repeller of a one-dimensional piecewise linear Markov map and μ is a Markov measure.

Fix an aperiodic 0-1 matrix $\mathbf{A}$. We define

$$\begin{aligned}\mathcal{H}(\mathbf{A}) = \{(f,\mu) \mid\ & f \text{ is a piecewise linear Markov map whose} \\ & \text{structure matrix is } \mathbf{A} \text{ and } \mu \text{ is a Markov} \\ & \text{measure on the repeller of } f \text{ satisfying } \alpha_{\min}^{(\mu)} < \alpha_{\max}^{(\mu)}\}\end{aligned}$$

and

$$\mathcal{X}(\mathbf{A}) = \{\mathcal{D}^{(\mu)} \mid (f,\mu) \in \mathcal{H}(\mathbf{A})\}.$$

For $\mathcal{D} \in \mathcal{X}(\mathbf{A})$, we set

$$C(\mathcal{D}) = \{(f,\mu) \in \mathcal{H}(\mathbf{A}) \mid \mathcal{D}^{(\mu)} = \mathcal{D}\}.$$

Definition 1.1. We say that $\mathcal{D} \in \mathcal{X}(\mathbf{A})$ has the *rigidity* if the following condition holds for any (f,μ), $(\widehat{f},\widehat{\mu}) \in C(\mathcal{D})$:

(D). There exists a homeomorphism $\zeta : \widehat{K} \to K$ such that

$$f \circ \zeta = \zeta \circ \widehat{f}, \quad \widehat{\mu} = \mu \circ \zeta \quad \text{and} \quad |\widehat{f}'| = |f'| \circ \zeta,$$

where K and $\widehat{K}$ are the repellers of f and $\widehat{f}$, respectively.

In this paper, we treat the multifractal rigidity problem when $\mathbf{A}$ has dimension 2. Thus, $\mathbf{A}$ is one of the following three:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1.2}$$

This rigidity was considered in some special cases, in [2] and [3]. We explain their results in Chapter 3.

The main results of this paper are the following two theorems, which give a complete characterization of spectra with the rigidity when $\mathbf{A}$ has dimension 2:

Theorem 1.1. *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. $\mathcal{D} \in \mathcal{X}(\mathbf{A})$ *has the rigidity if and only if* $\mathcal{D}$ *dose not coincide with the Legendre transform of the function*

$$\mathbb{R} \ni q \mapsto \log_r(\lambda^q + (1-\lambda)^q) \in \mathbb{R}$$

for any $\lambda \in (0, 1/2)$ *and* $r \in (0, 1)$.

Theorem 1.2. *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *Any* $\mathcal{D} \in \mathcal{X}(\mathbf{A})$ *has the rigidity.*

These theorems are proved in Chapter 5. Theorem 1.1 is a corollary of a theorem which contains the determination of the Markov measures corresponding to exceptional spectra.

Thermodynamic formalism tells us that both of $\alpha^{(\mu)}_{\min}, \alpha^{(\mu)}_{\max}$ are finite and there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ and a one-parameter family of measures $\{\mu_q\}_{q\in\mathbb{R}}$ such that $\beta' : \mathbb{R} \to (\alpha^{(\mu)}_{\min}, \alpha^{(\mu)}_{\max})$ is a decreasing diffeomorphism and $\dim_H \mu_q = \mathcal{D}^{(\mu)}(\alpha)$ for each $q \in \mathbb{R}$ with $\alpha = \beta'(q)$, where $\dim_H \nu$ denotes the Hausdorff dimension of the measure ν. The parameter q is an analogue of the inverse temperature in statistical physics, and a zero-temperature limit problem appears when we discuss the dimension spectrum at $\alpha^{(\mu)}_{\min} = \lim_{q\to+\infty} \beta'(q)$ or $\alpha^{(\mu)}_{\max} = \lim_{q\to-\infty} \beta'(q)$. In particular the problem whether $\mathcal{D}^{(\mu)}(\alpha^{(\mu)}_{\min}) = \mathcal{D}^{(\mu)}(\alpha^{(\mu)}_{\max}) = 0$ holds or not is a key problem.

This paper consists of five chapters. Chapter 2 is devoted to the definitions of some terms in this introduction and the description of the multifractal formalism for equilibrium measures via thermodynamic formalism. In Chapter 3 we introduce the results in [2] and [3]. An analysis at temperature zero is carried out in Chapter 4. We establish the multifractal formalism at temperature zero and give a simple condition for $\mathcal{D}^{(\mu)}(\alpha^{(\mu)}_{\min}) = \mathcal{D}^{(\mu)}(\alpha^{(\mu)}_{\max}) = 0$ in this chapter. Our main results are proved in Chapter 5.

2. Preliminaries

2.1. *One-dimensional Markov maps*

Let $N \geq 2$ be an integer and $f : \bigcup_{i=1}^N \Delta_i \to [0, 1]$ a $C^{1+\alpha}$ map, where $\Delta_1, ..., \Delta_N \subset [0, 1]$ are nondegenerate and disjoint closed intervals. The $N \times N$ matrix $\mathbf{A} = (A(ij))$ defined by

$$A(ij) = \begin{cases} 1 & (\Delta_i \cap f^{-1}\Delta_j \neq \emptyset), \\ 0 & (\Delta_i \cap f^{-1}\Delta_j = \emptyset) \end{cases}$$

is called the *structure matrix* of f.

Definition 2.1. (i) f is called a *one-dimensional Markov map* (with N branches) if the following three hold:

(a) If $\Delta_i \cap f^{-1}\Delta_j \neq \emptyset$ then $f(\Delta_i) \supset \Delta_j$ for all $i, j = 1, ..., N$.

(b) At least one entry in each row and column of $\mathbf{A}$ is equal to 1.

(c) $|f'| > 1$ on $\bigcup_{i=1}^N \Delta_i$.

(ii) A one-dimensional Markov map f is said to be *piecewise linear* if f' is constant on each Δ_i.

For a one-dimensional Markov map f, the set

$$K = \bigcap_{n=0}^{\infty} f^{-n}\left(\bigcup_{i=1}^{N} \Delta_i\right)$$

is called the *repeller* of f. We can define the *coding map* $\chi : \Sigma_{\mathbf{A}}^+ \to K$ by

$$\omega \to \bigcap_{n=0}^{\infty} f^{-n}\Delta_{\omega_{n+1}},$$

where

$$\Sigma_{\mathbf{A}}^+ = \{\omega = (\omega_n)_{n=1}^{\infty} \in \{1, ..., N\}^{\mathbb{N}} \mid A(\omega_n\omega_{n+1}) = 1 \text{ for all } n \geq 1\}.$$

We equip $\Sigma_{\mathbf{A}}^+$ with the product topology and define the *shift map* $\Sigma_{\mathbf{A}}^+ \to \Sigma_{\mathbf{A}}^+$ by

$$\sigma_{\mathbf{A}}(\omega_1\omega_2\cdots) = \omega_2\omega_3\cdots .$$

Then $\sigma_{\mathbf{A}}$ is a continuous mapping and the dynamical system $(\Sigma_{\mathbf{A}}^+, \sigma_{\mathbf{A}})$ is topologically conjugate to (K, f) by χ, i.e. χ is a homeomorphism such that

$$f \circ \chi = \chi \circ \sigma_{\mathbf{A}}.$$

2.2. *Multifractal formalism for equilibrium measures*

Let $f : \bigcup_{i=1}^N \Delta_i \to [0,1]$ be a one-dimensional Markov map with repeller K. We assume that the structure matrix $\mathbf{A}$ of f is *aperiodic*, i.e. all entries of $\mathbf{A}^k$ are positive for some positive integer k. For a continuous function $\phi : K \to \mathbb{R}$, we set

$$P(\phi) = \sup_{\mu}\left(h_\mu(f) + \int_K \phi \, d\mu\right), \tag{2.1}$$

where the $\sup_\mu$ is taken over all f-invariant Borel probability measures μ on K and $h_\mu(f)$ is the Kolmogorov-Sinai entropy of the measure μ. We call $P(\phi)$ the *pressure* of ϕ.

Definition 2.2. An f-invariant Borel probability measure μ on K is called an *equilibrium measure* for ϕ if μ attains the sup in (2.1), that is, $P(\phi) = h_\mu(f) + \int_K \phi \, d\mu$.

Let $\phi : K \to \mathbb{R}$ be a Hölder continuous function with $P(\phi) = 0$. There exists exactly one equilibrium measure μ for ϕ and we shall describe the dimension spectra of μ. To this end, we need the concept of Legendre transformation.

Definition 2.3. For a function $\beta : \mathbb{R} \to \mathbb{R}$, we define the function $\beta^* : \mathbb{R} \to [-\infty, +\infty)$ by

$$\beta^*(\alpha) = \inf_{q\in\mathbb{R}} \{q\alpha - \beta(q)\}.$$

β^* is called the *Legendre transform* of β.

We can define the function $\beta : \mathbb{R} \to \mathbb{R}$ by

$$P(q\phi + \beta(q) \log |f'|) = 0. \tag{2.2}$$

Theorem 2.1 ([4, 10, 11]). *Set* $K_\alpha = \{x \in K \mid d_\mu(x) = \alpha\}$ *and*

$$\alpha_{\min} = \inf_\nu -\frac{\int \phi \, d\nu}{\int \log |f'| \, d\nu}, \quad \alpha_{\max} = \sup_\nu -\frac{\int \phi \, d\nu}{\int \log |f'| \, d\nu}, \tag{2.3}$$

where both $\inf_\nu$ *and* $\sup_\nu$ *are taken over all* f*-invariant Borel probability measures* ν *on* K*. We have the following:*

(i) β *is strictly increasing, concave and real analytic.*

(ii) $\beta'(q) \to \alpha_{\min}$ $(q \to +\infty)$, $\beta'(q) \to \alpha_{\max}$ $(q \to -\infty)$.

(iii) $K_\alpha \neq \emptyset$ *if and only if* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

(iv) $\alpha_{\min} = \alpha_{\max}$ *if and only if* μ *is the equilibrium measure for the function* $(-\dim_H K) \log |f'|$*, in which case*

$$\mathcal{D}^{(\mu)}(\alpha_{\min}) = \dim_H K.$$

(v) If $\alpha_{\min} < \alpha_{\max}$ *then*

$$\mathcal{D}^{(\mu)}(\alpha) = \beta^*(\alpha) \quad \textit{for all} \quad \alpha \in (\alpha_{\min}, \alpha_{\max}).$$

In particular, $\mathcal{D}^{(\mu)}$ *is strictly concave in* $(\alpha_{\min}, \alpha_{\max})$*. Furthermore we have*

$$\max_{\alpha\in(\alpha_{\min},\, \alpha_{\max})} \mathcal{D}^{(\mu)}(\alpha) = -\beta(0) = \dim_H K.$$

2.3. *Markov measures*

Let $f : \bigcup_{i=1}^N \Delta_i \to [0, 1]$ be a one-dimensional Markov map with repeller K and aperiodic structure matrix $\mathbf{A}$. An element of $\bigcup_{n=1}^\infty \{1, ..., N\}^n$ is called a *word* and we write $|w| = n$ for each word $w \in \{1, ..., N\}^n$. For an $N \times N$ matrix $\mathbf{B} = (B(ij))$ and a word $w = w_1 \cdots w_{|w|}$ with $|w| \geq 2$, we write

$$B(w) = B(w_1w_2)B(w_2w_3) \cdots B(w_{|w|-1}w_{|w|}).$$

A word w is said to be $\mathbf{A}$-*admissible* if $|w| \geq 2$ and $A(w_kw_{k+1}) = 1$ for each $k = 1, ..., |w| - 1$. For each $\mathbf{A}$-admissible word w, we set

$$\Delta_w = \bigcap_{k=0}^{|w|-1} f^{-k}\Delta_{w_{k+1}}.$$

We define a natural class of equilibrium measures. Let μ be a Borel probability measure on K.

Definition 2.4. (i) μ is called a *Markov measure* if there exists an $N \times N$ real matrix $\mathbf{P} = (P(ij))$ satisfies the following properties:

(a) $\mathbf{P}$ is a *stochastic matrix*, i.e. all entries of $\mathbf{P}$ are nonnegative and $\sum_{j=1}^{N} P(ij) = 1$ for each $i = 1, ..., N$.

(b) $P(ij) > 0$ if and only if $A(ij) = 1$ for each $i, j = 1, ..., N$.

(c) For each $\mathbf{A}$-admissible word w, we have

$$\mu(\Delta_w \cap K) = p_{w_1} P(w), \tag{2.4}$$

where $\mathbf{p} = (p_1, ..., p_N) \in \mathbb{R}^N$ is the normalized left Perron-Frobenius eigenvector for $\mathbf{P}$.

(ii) A Markov measure μ is called a *Bernoulli measure* if all entries of $\mathbf{A}$ are equal to 1 and $\mathbf{P}$ is a matrix of the form $\begin{pmatrix} b_1 & \cdots & b_N \\ & \cdots & \\ b_1 & \cdots & b_N \end{pmatrix}$ called a *Bernoulli matrix*, where $b_i > 0$ and $b_1 + \cdots + b_N = 1$.

(iii) We define a Markov measure on $\Sigma_{\mathbf{A}}^+$ by replacing $\Delta_w \cap K$ with $[w]_{\mathbf{A}}$ in (2.4), where

$$[w]_{\mathbf{A}} = \{\omega \in \Sigma_{\mathbf{A}}^+ \mid \omega_k = w_k \text{ for all } 1 \leq k \leq |w|\}.$$

If μ is the Markov measure corresponding to a stochastic matrix $\mathbf{P} = (P(ij))$, then μ is the unique equilibrium measure for $\phi : K \to \mathbb{R}$ defined by $\phi|_{\Delta_{ij} \cap K} = \log P(ij)$. Moreover we have

$$h_\mu(f) = \sum_{i,j} -p_i P(ij) \log P(ij),$$

where we put $0 \log 0 = 0$.

Assume that f is a piecewise linear Markov map with derivatives $r_i = 1/|f'|_{\Delta_i}$ $(i = 1, ..., N)$ and μ is the Markov measure corresponding to $\mathbf{P} = (P(ij))_{1 \leq i,j \leq N}$. We can describe the multifractal formalism by using matrices. Indeed, we can easily check that $\beta(q)$ in (2.2) is the unique real number β such that

$$\text{the spectral radius of the matrix } (P(ij)^q r_j^{-\beta}) \text{ is equal to 1.}$$

In particular, if all entries of $\mathbf{A}$ are equal to 1 and $\mathbf{P}$ is the Bernoulli matrix $\begin{pmatrix} b_1 & \cdots & b_N \\ & \cdots & \\ b_1 & \cdots & b_N \end{pmatrix}$ then $\beta(q)$ is the unique real number β such that

$$b_1^q r_1^{-\beta} + \cdots + b_N^q r_N^{-\beta} = 1.$$

In addition, we can obtain the following simpler representation of $\alpha_{\min}, \alpha_{\max}$. We need this representation in Chapter 4. For a word w with $|w| \geq 2$, we write

$$r(w) = r_{w_2} \cdots r_{w_{|w|}}.$$

A word w with $|w| \geq 2$ is called a *cycle* if $w_1 = w_{|w|}$. A cycle w is said to be *simple* if $w_i \neq w_j$ for any $1 \leq i \neq j \leq |w| - 1$. We set

$$S = \{w \mid w \text{ is an } \mathbf{A}\text{-admissible simple cycle}\}.$$

By Theorem 2.1 (iii), we can easily show that

$$\alpha_{\min} = \min_{w \in S} \frac{\log P(w)}{\log r(w)}, \quad \alpha_{\max} = \max_{w \in S} \frac{\log P(w)}{\log r(w)}. \tag{2.5}$$

In particular, if all entries of $\mathbf{A}$ are equal to 1 and $\mathbf{P}$ is the Bernoulli matrix $\left(\begin{smallmatrix} b_1 & \cdots & b_N \\ & \vdots & \\ b_1 & \cdots & b_N \end{smallmatrix}\right)$ then

$$\alpha_{\min} = \min_{1 \le i \le N} \frac{\log b_i}{\log r_i}, \quad \alpha_{\max} = \max_{1 \le i \le N} \frac{\log b_i}{\log r_i}.$$

3. Known Results on Multifractal Rigidity

We mention the result of Barreira, Pesin and Schmeling, in [2]. Assume that $\mathbf{A} = \left(\begin{smallmatrix} 1 & \cdots & 1 \\ & \vdots & \\ 1 & \cdots & 1 \end{smallmatrix}\right)$. We define

$$\mathcal{H}^B(\mathbf{A}) = \{(f, \mu) \in \mathcal{H}(\mathbf{A}) \mid \mu \text{ is a Bernoulli measure}\}$$

and

$$\mathcal{X}^B(\mathbf{A}) = \{\mathcal{D}^{(\mu)} \mid (f, \mu) \in \mathcal{H}^B(\mathbf{A})\}.$$

For $\mathcal{D} \in \mathcal{X}^B(\mathbf{A})$, we set

$$C^B(\mathcal{D}) = \{(f, \mu) \in \mathcal{H}^B(\mathbf{A}) \mid \mathcal{D}^{(\mu)} = \mathcal{D}\}.$$

Definition 3.1. We say that $\mathcal{D} \in \mathcal{X}^B(\mathbf{A})$ has *B-rigidity* if the condition (D) in the introduction holds for any (f, μ), $(\widehat{f}, \widehat{\mu}) \in C^B(\mathcal{D})$.

Theorem 3.1 ([2]). *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *Any* $\mathcal{D} \in \mathcal{X}^B(\mathbf{A})$ *has B-rigidity.*

Next we mention the results of Barreira and Saravia, in [3]. Let $\mathbf{A}$ be an $N \times N$ aperiodic matrix such that each entry is 0 or 1 and μ a $\sigma_{\mathbf{A}}$-invariant Borel probability measure on $\Sigma_{\mathbf{A}}^+$. We set

$$E_\alpha = \left\{ \omega \in \Sigma_{\mathbf{A}}^+ \mid \lim_{n \to \infty} -\frac{\log \mu([\omega_1 \cdots \omega_n]_{\mathbf{A}})}{n} = \alpha \right\}$$

and

$$\alpha_{\min}^{(\mu)} = \inf\{\alpha \mid E_\alpha \neq \emptyset\}, \quad \alpha_{\max}^{(\mu)} = \sup\{\alpha \mid E_\alpha \neq \emptyset\}.$$

We define the function $\mathcal{E}^{(\mu)} : [\alpha_{\min}^{(\mu)}, \alpha_{\max}^{(\mu)}] \to \mathbb{R}$ by

$$\mathcal{E}^{(\mu)}(\alpha) = h(\sigma_{\mathbf{A}} | E_\alpha),$$

where $h(\sigma_{\mathbf{A}}|Z)$ denotes the topological entropy of $Z \subset \Sigma_{\mathbf{A}}^+$ (Z need not be compact nor $\sigma_{\mathbf{A}}$-invariant). We call $\mathcal{E}^{(\mu)}$ the *entropy spectrum* of μ.

We define

$$\mathcal{X}_E(\mathbf{A}) = \{\mathcal{E}^{(\mu)} \mid \mu \text{ is a Markov measure on } \Sigma_{\mathbf{A}}^+ \text{ with } \alpha_{\min}^{(\mu)} < \alpha_{\max}^{(\mu)}\}.$$

A homeomorphism $\rho : \Sigma_{\mathbf{A}}^{+} \to \Sigma_{\mathbf{A}}^{+}$ is called an *automorphism* on $\Sigma_{\mathbf{A}}^{+}$ if $\sigma_{\mathbf{A}} \circ \rho = \rho \circ \sigma_{\mathbf{A}}$. We denote by $\mathrm{Aut}(\Sigma_{\mathbf{A}}^{+})$ the set of all automorphisms on $\Sigma_{\mathbf{A}}^{+}$.

Definition 3.2. We say that $\mathcal{E} \in \mathcal{X}_E(\mathbf{A})$ has the *rigidity* if the following condition holds for any Markov measures $\mu, \widehat{\mu}$ on $\Sigma_{\mathbf{A}}^{+}$ with $\mathcal{E}^{(\mu)} = \mathcal{E}^{(\widehat{\mu})} = \mathcal{E}$:

(E). There exists $\rho \in \mathrm{Aut}(\Sigma_{\mathbf{A}}^{+})$ such that

$$\widehat{\mu} = \mu \circ \rho.$$

We use Σ_N^{+} instead of $\Sigma_{\mathbf{A}}^{+}$ when all entries of $\mathbf{A}$ are equal to 1. Hedlund proved the following theorem in [6]. This theorem is not displayed explicitly in [6], and we recommend Kitchens' book [8] to the readers for the proof.

Theorem 3.2 ([6])**.** $\mathrm{Aut}(\Sigma_2^{+}) = \{\mathrm{id}, \mathrm{flip}\}$. *Where* $\mathrm{flip} : \Sigma_2^{+} \to \Sigma_2^{+}$ *is defined by* $\omega \mapsto \overline{\omega},\ \overline{\omega}_n = \begin{cases} 1 & (\omega_n = 2) \\ 2 & (\omega_n = 1) \end{cases}$.

The multifractal rigidities proved in [3] are the following:

Theorem 3.3 ([3])**.** *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *Let* $\mathcal{E} \in \mathcal{X}_E(\mathbf{A})$. *We have the following:*

(i) If $\mathcal{E} \neq (-\log(\lambda^q + (1-\lambda)^q))^*$ *for any* $\lambda \in (0, 1/2)$ *then* $\mathcal{E}$ *has the rigidity.*

(ii) Assume that $\mathcal{E} = (-\log(\lambda^q + (1-\lambda)^q))^*$ *for some* $\lambda \in (0, 1/2)$. *Let* $\mu, \widehat{\mu}$ *be Markov measures on* $\Sigma_{\mathbf{A}}^{+}$ *such that* $\mathcal{E}^{(\mu)} = \mathcal{E}^{(\widehat{\mu})} = \mathcal{E}$. *Then the following hold:*

(a) μ *corresponds to the following four matrices:*

$$\begin{pmatrix} 1-\lambda & \lambda \\ 1-\lambda & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}, \begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix}.$$

(b) (E) holds with $\rho = \mathrm{id}$ *if and only if both* μ *and* $\widehat{\mu}$ *correspond to the same matrix in (a). (E) holds with* $\rho = \mathrm{flip}$ *if and only if either* $(\mu, \widehat{\mu})$ *or* $(\widehat{\mu}, \mu)$ *corresponds to* $\left(\left(\begin{smallmatrix} 1-\lambda & \lambda \\ 1-\lambda & \lambda \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{smallmatrix}\right)\right)$.

Theorem 3.4 ([3])**.** *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *Any* $\mathcal{E} \in \mathcal{X}_E(\mathbf{A})$ *has the rigidity.*

We explain the relation between the results in [3] and ours. For $\theta \in (0, 1)$, we define the distance d_θ on $\Sigma_{\mathbf{A}}^{+}$ by

$$d_\theta(\omega, \omega') = \theta^{\sup\{n \geq 1 \mid \omega_1 = \omega'_1, \ldots, \omega_n = \omega'_n\}}.$$

The product topology of $\Sigma_{\mathbf{A}}^{+}$ coincides with the topology induced by d_θ.

[2] established the relation between the entropy spectra and the dimension spectra for a shift-invariant measure.

Theorem 3.5 ([2])**.** *For each* $\alpha \in (-\infty, +\infty)$ *and* $\theta \in (0, 1)$, *we have*

$$\mathcal{E}^{(\mu)}(\alpha) = \mathcal{D}^{(\mu)}(\alpha / \log \theta^{-1}) \cdot \log \theta^{-1},$$

where we equip $\Sigma_{\mathbf{A}}^{+}$ *with the distance* d_θ.

Let $f : \bigcup_{i=1}^{N} \Delta_i \to [0,1]$ be piecewise linear. If $|f'|$ is constant on $\bigcup_{i=1}^{N} \Delta_i$ and $\theta = 1/|f'|$ then the coding map $\chi : \Sigma_{\mathbf{A}}^{+} \to K$ is a bi-Lipschitz continuous mapping. Thus, Theorem 3.5 provides us with a translation rule the multifractal rigidity problem based on Definition 3.2 to ours, namely we replace $\mathcal{H}(\mathbf{A}), \mathcal{X}(\mathbf{A}), C(\mathcal{D})$ with

$$\begin{aligned}
\mathcal{H}^c(\mathbf{A}) &= \{(f,\mu) \in \mathcal{H}(\mathbf{A}) \mid |f'| \text{ is constant on } \textstyle\bigcup_{i=1}^{N} \Delta_i\}, \\
\mathcal{X}^c(\mathbf{A}) &= \{\mathcal{D}^{(\mu)} \mid (f,\mu) \in \mathcal{H}^c(\mathbf{A})\}, \\
C^c(\mathcal{D}) &= \{(f,\mu) \in \mathcal{H}^c(\mathbf{A}) \mid \mathcal{D}^{(\mu)} = \mathcal{D}\},
\end{aligned}$$

respectively. Theorem 3.3 and 3.4 give a complete answers to this problem.

4. Multifractal Analysis at Temperature Zero

4.1. *Zero-temperature limit measures*

Let $f : \bigcup_{i=1}^{N} \Delta_i \to [0,1]$ be a one-dimensional Markov map with topologically mixing repeller K and μ the equilibrium measures for a Hölder continuous function $\phi : K \to \mathbb{R}$ with $P(\phi) = 0$. In this chapter, we consider the multifractal formalism at the endpoints $\alpha_{\min}, \alpha_{\max}$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be the same as that in (2.2).

Fix $q \in \mathbb{R}$. We denote by μ_q the equilibrium measure for $q\phi + \beta(q)\log|f'|$.

Lemma 4.1 ([10]). *Put $\alpha = \beta'(q)$.*

(i) $\mu_q(K_\alpha) = 1$ and $d_{\mu_q}(x) = \beta^*(\alpha)$ for all $x \in K_\alpha$.

(ii) We have

$$\beta^*(\alpha) = \frac{h_{\mu_q}(f)}{\int \log|f'|\, d\mu_q}.$$

The relation between $\mathcal{D}^{(\mu)}$ and β^* in Theorem 2.1 follows from this lemma.

The parameter q is an analogue of the inverse temperature in statistical physics. We call an accumulation point of the family of measures $\{\mu_q\}_{q\in\mathbb{R}}$ when $q \to +\infty$ or $-\infty$ a *zero-temperature limit measure* (we equip the space of measures with the weak* topology). We set

$$\begin{aligned}
\mathcal{M}_\infty^{+} &= \{\text{zero-temperature limit measures when } q \to +\infty\}, \\
\mathcal{M}_\infty^{-} &= \{\text{zero-temperature limit measures when } q \to -\infty\}.
\end{aligned}$$

Zero-temperature limit measures play an important role in the next two sections.

The following is an immediate consequence of the upper semicontinuity of β^*.

Proposition 4.1. *Both of $\beta^*(\alpha_{\min})$ and $\beta^*(\alpha_{\max})$ are finite and*

$$\begin{aligned}
\beta^*(\alpha_{\min}) &= \lim_{q\to+\infty}\{q\alpha_{\min} - \beta(q)\} = \lim_{\alpha\downarrow\alpha_{\min}} \beta^*(\alpha), \\
\beta^*(\alpha_{\max}) &= \lim_{q\to-\infty}\{q\alpha_{\max} - \beta(q)\} = \lim_{\alpha\uparrow\alpha_{\max}} \beta^*(\alpha).
\end{aligned}$$

4.2. *Multifractal formalism at temperature zero*

We restrict ourselves to the case where f is piecewise linear and μ is the Markov measure corresponding to a stochastic matrix $\mathbf{P} = (P(ij))$. We write

$$r_i = 1/|f'|_{\Delta_i} \quad (i = 1, ..., N).$$

The aim of this section is to show that the multifractal formalism in Theorem 2.1 holds at endpoints, or the following proposition holds:

Proposition 4.2. $\mathcal{D}^{(\mu)}(\alpha_{\min}) = \beta^*(\alpha_{\min})$ *and* $\mathcal{D}^{(\mu)}(\alpha_{\max}) = \beta^*(\alpha_{\max})$.

The following lemma, which is an analogue of Lemma 4.1, is essential for the proof of Proposition 4.2.

Lemma 4.2. *Take* $\mu_\infty \in \mathcal{M}^+_\infty$.
(i) $\operatorname{supp}\mu_\infty \subset K_{\alpha_{\min}}$ *and* $d_{\mu_\infty}(x) = \beta^*(\alpha_{\min})$ *for all* $x \in \operatorname{supp}\mu_\infty$.
(ii) We have

$$\beta^*(\alpha_{\min}) = \frac{h_{\mu_\infty}(f)}{\int \log|f'|\, d\mu_\infty}.$$

An analogous result holds for $\alpha_{\max}$ *and* $\mu_\infty \in \mathcal{M}^-_\infty$.

Proof. Take $q \in \mathbb{R}$. By the Perron-Frobenius theorem, we can find a right eigenvector $\mathbf{a}_q = {}^t(a_{q,1}, ..., a_{q,N}) \in \mathbb{R}^N$ such that $(P(ij)^q r_j^{-\beta(q)})\mathbf{a}_q = \mathbf{a}_q$ and all entries of $\mathbf{a}_q$ are positive. We define a stochastic matrix $\mathbf{B}_q$ by $B_q(ij) = a_{q,i}^{-1} P(ij)^q r_j^{-\beta(q)} a_{q,j}$. Again by the Perron-Frobenius theorem, we can find the unique stochastic vector $\mathbf{b}_q = (b_{q,1}, ..., b_{q,N}) \in \mathbb{R}^N$ such that $\mathbf{b}_q\mathbf{B}_q = \mathbf{b}_q$. Then μ_q is the Markov measure corresponding to the stochastic matrix $\mathbf{B}_q$.

Take $\mu_\infty \in \mathcal{M}^+_\infty$. There exists a stochastic matrix $\mathbf{B}$, a stochastic vector $\mathbf{b}$ and a sequence $\{q_n\} \subset \mathbb{R}$ such that $q_n \to +\infty$, $\mathbf{B}_{q_n} \to \mathbf{B}$, $\mathbf{b}_{q_n} \to \mathbf{b}$ $(n \to \infty)$ and μ_∞ is the Markov measure corresponding to $\mathbf{B}$ and $\mathbf{b}$.

We prove (i). For any $\mathbf{A}$-admissible cycle w, we obtain that $\frac{\log P(w)}{\log r(w)} \geq \alpha_{\min}$ by (2.5). Moreover since $B_q(w) = P(w)^q r(w)^{-\beta(q)} = r(w)^{q\frac{\log P(w)}{\log r(w)} - \beta(q)}$, we observe by Proposition 4.1 that

$$B(w) = \lim_{n\to\infty} B_{q_n}(w) = \begin{cases} r(w)^{\beta^*(\alpha_{\min})} > 0 & \left(\frac{\log P(w)}{\log r(w)} = \alpha_{\min}\right), \\ 0 & \left(\frac{\log P(w)}{\log r(w)} > \alpha_{\min}\right). \end{cases} \tag{4.1}$$

Fix $x \in \operatorname{supp}\mu_\infty$ and put $\omega = \chi^{-1}(x)$. There exists a sequence of integers $1 \leq n_1 < n_2 < \cdots$ such that $\omega_{n_1} = \omega_{n_2} = \cdots$. We have $B(\omega_{n_k} \cdots \omega_{n_{k+1}}) > 0$ since $x \in \operatorname{supp}\mu_\infty$, and thus, by (4.1), we obtain $\frac{\log P(\omega_{n_k} \cdots \omega_{n_{k+1}})}{\log r(\omega_{n_k} \cdots \omega_{n_{k+1}})} = \alpha_{\min}$ for all $k \geq 1$. This implies that $\lim_{n\to\infty} \frac{\log P(\omega_1 \cdots \omega_n)}{\log r(\omega_1 \cdots \omega_n)} = \alpha_{\min}$, that is, $x \in K_{\alpha_{\min}}$. Moreover we obtain $B(\omega_{n_k} \cdots \omega_{n_{k+1}}) = r(\omega_{n_k} \cdots \omega_{n_{k+1}})^{\beta^*(\alpha_{\min})}$ for all $k \geq 1$. This implies that $\lim_{n\to\infty} \frac{\log B(\omega_1 \cdots \omega_n)}{\log r(\omega_1 \cdots \omega_n)} = \beta^*(\alpha_{\min})$, that is, $d_{\mu_\infty}(x) = \beta^*(\alpha_{\min})$.

We prove (ii). Put $\alpha_n = \beta'(q_n)$. We have

$$\begin{aligned} h_{\mu_{q_n}}(f) &= \sum_{i,j} -b_{q_n,i} B_{q_n}(ij) \log B_{q_n}(ij) \\ &\to \sum_{i,j} -b_i B(ij) \log B(ij) = h_{\mu_\infty}(f) \quad (n \to \infty), \end{aligned}$$

and hence, by Lemma 4.1 (ii), we have

$$\beta^*(\alpha_n) \to \frac{h_{\mu_\infty}(f)}{\int \log |f'| \, d\mu_\infty} \quad (n \to \infty).$$

On the other hand, by Proposition 4.1, we have $\beta^*(\alpha_n) \to \beta^*(\alpha_{\min})$ $(n \to \infty)$. □

Proof of Proposition 4.2. We only show that $\mathcal{D}^{(\mu)}(\alpha_{\min}) = \beta^*(\alpha_{\min})$.

$\mathcal{D}^{(\mu)}(\alpha_{\min}) \geq \beta^*(\alpha_{\min})$ follows from Lemma 4.2 (i) and the mass distribution principle. We will show that $\mathcal{D}^{(\mu)}(\alpha_{\min}) \leq \beta^*(\alpha_{\min})$.

Fix $q \in \mathbb{R}$. It is easy to check that

$$K_{\alpha_{\min}} \subset \{x \in K \mid d_{\mu_q}(x) = q\alpha_{\min} - \beta(q)\},$$

and thus, we have $\mathcal{D}^{(\mu)}(\alpha_{\min}) \leq q\alpha_{\min} - \beta(q)$. Since $q \in \mathbb{R}$ is arbitrary, we obtain $\mathcal{D}^{(\mu)}(\alpha_{\min}) \leq \beta^*(\alpha_{\min})$. □

It seems that many researchers on multifractal analysis believe that Proposition 4.2 holds for any one-dimensional Markov map (need not be piecewise linear) and the equilibrium measure for a Hölder continuous function (need not be a Markov measure). However the present author could not find the proof in literature.

4.3. *Nondegeneracy of spectra*

The concept of nondegeneracy first appeared in [11].

Definition 4.1. The spectrum $\mathcal{D}^{(\mu)}$ is said to be *nondegenerate* if $\mathcal{D}^{(\mu)}(\alpha_{\min}) = \mathcal{D}^{(\mu)}(\alpha_{\max}) = 0$ holds.

We write $\phi \sim \psi$ $(q \to +\infty)$ for two functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ with $\phi, \psi > 0$ if $\phi(q)/\psi(q) \to 1 (q \to +\infty)$ holds. The following is an easy but important consequence from nondegeneracy and works essentially in the next chapter.

Lemma 4.3. *Let $a, b, r > 0, b \neq 1$ and β the same as that in (2.2).*

(i) If $\mathcal{D}^{(\mu)}(\alpha_{\min}) = 0$ then

$$\lim_{q \to +\infty} \frac{1 - a^{q\alpha_{\min} - \beta(q)}}{1 - b^{q\alpha_{\min} - \beta(q)}} = \frac{\log a}{\log b}$$

and

$$r^{qa - \beta(q)} \sim (r^{a - \alpha_{\min}})^q \quad (q \to +\infty).$$

(ii) If $\mathcal{D}^{(\mu)}(\alpha_{\max}) = 0$ *then*

$$\lim_{q\to-\infty} \frac{1-a^{q\alpha_{\max}-\beta(q)}}{1-b^{q\alpha_{\max}-\beta(q)}} = \frac{\log a}{\log b}.$$

and

$$r^{qa-\beta(q)} \sim (r^{a-\alpha_{\max}})^q \quad (q\to-\infty).$$

Proof. We only discuss (i). We have $\lim_{q\to+\infty}(q\alpha_{\min}-\beta(q)) = 0$ from Proposition 4.1. Thus, the first equation follows from l'Hôpital's lemma and the second one follows from $r^{qa-\beta(q)} = (r^{a-\alpha_{\min}})^q \times r^{q\alpha_{\min}-\beta(q)}$, immediately. □

Let $\mathbf{B}$ be an $N\times N$ nonnegative matrix. $\mathbf{B}$ is said to be *irreducible* if for each $i,j = 1,...,N$ there exists a positive integer k such that $B^k(ij) > 0$. $\mathbf{B}$ is called a *permutation matrix* if exactly one entry is 1 and any other entry is 0 for each row and column of $\mathbf{B}$. $\sigma \mapsto (\delta_{\sigma(i)j})_{1\le i,j\le N}$ is a one-to-one correspondence between the symmetric group of N-words $\mathfrak{S}_N$ and the set of all $N\times N$ permutation matrices. An $N\times N$ permutation matrix is said to be *cyclic* if it corresponds to a cyclic permutation with length N. For two words w, w' we write $w\cap w' \neq \emptyset$ if $\{w_1,...,w_{|w|}\}\cap\{w'_1,...,w'_{|w'|}\} \neq \emptyset$. We define the cycle $\mathrm{rot}(w)$ for a cycle w by

$$\mathrm{rot}(w) = w_2 w_3 \cdots w_{|w|} w_2.$$

For two cycles w, w' we write $w \sim_{\mathrm{rot}} w'$ if

$$\text{there exists an integer } n\ge 1 \text{ such that } \mathrm{rot}^n(w) = w'.$$

Proposition 4.3. *Let* $\mathbf{B}$ *be an irreducible stochastic matrix. Then the following three are equivalent:*

(i) Each entry of $\mathbf{B}$ *is* 0 *or* 1.

(ii) $\mathbf{B}$ *is a cyclic permutation matrix.*

(iii) $w \sim_{\mathrm{rot}} w'$ *holds for any two* $\mathbf{B}$*-admissible simple cycles* w, w' *with* $w\cap w' \neq \emptyset$.

Proof. It is easy to see that (i) implies (ii) and (ii) implies (iii). We show that (iii) implies (i). If (i) does not hold then there exist $i_1, i_2, i_3 \in \{1,...,N\}$ with $i_2 \neq i_3$ such that $B(i_1 i_2) > 0$ and $B(i_1 i_3) > 0$. By irreducibility, we can take $\mathbf{B}$-admissible words w, w' such that $w_1 = i_2$, $w'_1 = i_3$, $w_{|w|} = w'_{|w'|} = i_1$ and $w_i \neq w_j$, $w'_i \neq w'_j$ if $i\neq j$. Clearly, $i_1 w$, $i_1 w'$ are $\mathbf{B}$-admissible simple cycles with $i_1 w \cap i_1 w' \neq \emptyset$ and there exists no integer $n \ge 1$ such that $\mathrm{rot}^n(i_1 w) = i_1 w'$. □

Recall that $\mathbf{A}$ is the structure matrix of f and S is the set consisting of all $\mathbf{A}$-admissible simple cycles. We set

$$\begin{aligned} S_{\min} =& \left\{ w\in S \mid \frac{\log P(w)}{\log r(w)} = \alpha_{\min} \right\}, \\ S_{\max} =& \left\{ w\in S \mid \frac{\log P(w)}{\log r(w)} = \alpha_{\max} \right\}. \end{aligned} \tag{4.2}$$

Note that if two **A**-admissible cycle w, w' satisfy $w \sim_{\mathrm{rot}} w'$ then $\frac{\log P(w)}{\log r(w)} = \frac{\log P(w')}{\log r(w')}$. The following is the main result of this section.

Theorem 4.1. *The following four are equivalent:*
(i) $\mathcal{D}^{(\mu)}(\alpha_{\min}) = 0$.
(ii) $h_{\mu_\infty}(f) = 0$ *for all* $\mu_\infty \in \mathcal{M}_\infty^+$.
(iii) $h_{\mu_\infty}(f) = 0$ *for some* $\mu_\infty \in \mathcal{M}_\infty^+$.
(iv) $w \sim_{\mathrm{rot}} w'$ *holds for any two* $w, w' \in S_{\min}$ *with* $w \cap w' \neq \emptyset$.
We have an analogous result for $\alpha_{\max}$ *by replacing* $\mathcal{M}_\infty^+$ *and* $S_{\min}$ *with* $\mathcal{M}_\infty^-$ *and* $S_{\max}$*, respectively.*

Proof. The equivalence of (i), (ii), (iii) follows from Lemma 4.2 (ii) immediately.

We show the equivalence of (i) and (iv). We use a technique in the study of Markov chains (see for example [7] for the details).

Fix $\mu_\infty \in \mathcal{M}_\infty^+$ and take a stochastic matrix $\mathbf{B}$, a stochastic vector $\mathbf{b}$ and a sequence $\{q_n\} \subset \mathbb{R}$ such that $q_n \to +\infty$, $\mathbf{B}_{q_n} \to \mathbf{B}$, $\mathbf{b}_{q_n} \to \mathbf{b}$ $(n \to \infty)$ and μ_∞ is the Markov measure corresponding to $\mathbf{B}$ and $\mathbf{b}$. For $i, j \in \{1, ..., N\}$ we write $i \leftrightarrow j$ if $i = j$ or there exists a $\mathbf{B}$-admissible cycle w such that $i, j \in \{w_1, ..., w_{|w|-1}\}$. $\leftrightarrow$ is an equivalence relation on $\{1, ..., N\}$ and for each equivalence class C the submatrix of $\mathbf{B}$ corresponding to C is an irreducible matrix. An equivalence class C is called an *ergodic set* if the corresponding submatrix is a stochastic matrix. We can show that at least one ergodic set exists and we can write

$$\mathbf{B} = \left(\begin{array}{ccc} \mathbf{B}^{(1)} & & \\ & \ddots & \mathbf{0} \\ \mathbf{0} & & \ddots \\ & & \mathbf{B}^{(e)} \\ \hline & * & Q \end{array} \right),$$

where each $\mathbf{B}^{(k)}$ is the submatrix corresponding to an ergodic set and Q is the submatrix corresponding to $\{1, ..., N\} \setminus \bigcup\{\text{ergodic set}\}$. Let $\mu_\infty^{(k)}$ be the Markov measure corresponding to the $\mathbf{B}^{(k)}$. There exist nonnegative numbers $\lambda^{(1)}, ..., \lambda^{(e)}$ such that $\sum_{k=1}^{e} \lambda^{(k)} = 1$ and $\mu_\infty = \sum_{k=1}^{e} \lambda^{(k)} \mu_\infty^{(k)}$. We have by (4.1) that

$$\bigcup_{k=1}^{e} \{\mathbf{B}^{(k)}\text{-admissible simple cycles}\} \subset S_{\min}. \tag{4.3}$$

Assume that (iv) holds. Then (4.3) shows that $w \sim_{\mathrm{rot}} w'$ holds for each $k = 1, ..., e$ and any two $\mathbf{B}^{(k)}$-admissible simple cycles w, w' with $w \cap w' \neq \emptyset$. Since $\mathbf{B}^{(k)}$ is an irreducible stochastic matrix, each entry of $\mathbf{B}^{(k)}$ is 0 or 1 from Proposition 4.3. This implies that $h_{\mu_\infty^{(k)}}(f) = 0$ for each $k = 1, ..., e$, and thus, we have $h_{\mu_\infty}(f) = \sum_k \lambda^{(k)} h_{\mu_\infty^{(k)}}(f) = 0$, that is, $\mathcal{D}^{(\mu)}(\alpha_{\min}) = 0$.

Assume that (i) holds. We will show that

$$B(w_1 w_2) = \cdots = B(w_{|w|-1} w_{|w|}) = 1 \text{ for any } w \in S_{\min}. \tag{4.4}$$

Take $w \in S_{\min}$ arbitrary. Then by (4.1) we have

$$\prod_{l=1}^{|w|-1} B(w_l w_{l+1}) = B(w) = r(w)^{\mathcal{D}^{(\mu)}(\alpha_{\min})} = 1. \tag{4.5}$$

Since $\mathbf{B}$ is a stochastic matrix, we observe that $0 \le B(ij) \le 1$ for each $1 \le i, j \le N$. Thus, (4.5) implies that $B(w_l w_{l+1}) = 1$ for each $l = 1, ..., |w| - 1$.

Fix $k \in \{1, ..., e\}$. We observe that each entry of $\mathbf{B}^{(k)}$ is 0 or 1 by (4.3), (4.4) and the irreducibility of $\mathbf{B}^{(k)}$. Thus, Proposition 4.3 tells us that $w \sim_{\mathrm{rot}} w'$ holds for any two $\mathbf{B}^{(k)}$-admissible simple cycles w, w' with $w \cap w' \neq \emptyset$. Therefore we complete the proof if we show that the opposite inclusion holds in (4.3).

Take $w \in S_{\min}$ arbitrary. Since $\mathbf{B}$ is a stochastic matrix, (4.4) implies that

$$B(w_l j) = \begin{cases} 1 & (j = w_{l+1}), \\ 0 & (j \neq w_{l+1}) \end{cases} \tag{4.6}$$

for each $l = 1, ..., |w| - 1$.

Let C be the equivalence class which contains w_1. We will show that $C = \{w_1, ..., w_{|w|-1}\}$. $C \supset \{w_1, ..., w_{|w|-1}\}$ is obvious. Take $i \in C$. There exists a $\mathbf{B}$-admissible word z such that $z_1 = w_1$ and $z_{|z|} = i$. We observe that $z_2 = w_2$ by (4.6). By induction we obtain $z_3 = w_3, z_4 = w_4, ...$ and it implies that $|z| \le |w|$ and $i = w_{|z|}$. Thus $C \subset \{w_1, ..., w_{|w|-1}\}$.

Let $\mathbf{B}_C$ be the submatrix of $\mathbf{B}$ corresponding to C. We know by (4.6) and $C = \{w_1, ..., w_{|w|-1}\}$ that $\mathbf{B}_C$ is a stochastic matrix, that is, C is an ergodic set. Therefore we have $w \in \bigcup_{k=1}^{e} \{\mathbf{B}^{(k)}\text{-admissible simple cycles}\}$. Since $w \in S_{\min}$ is arbitrary we obtain the opposite inclusion in (4.3). □

Example 4.1. We determine whether $\mathcal{D}^{(\mu)}(\alpha_{min}) = 0$ or $\mathcal{D}^{(\mu)}(\alpha_{min}) > 0$ for some μ in the case that $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ and $r_1 = r_3 = 1/3, r_2 = 1/9 = (1/3)^2$.

(i) If

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{9}{28} & \frac{1}{4} & \frac{3}{7} \end{pmatrix}.$$

then $\alpha_{\min} = \frac{\log 2}{\log 3}$ and $S_{\min} = \{11, 22, 232, 323\}$. Since $22 \cap 232 \neq \emptyset$ and $22 \not\sim_{\mathrm{rot}} 232$ therefore $\mathcal{D}^{(\mu)}(\alpha_{\min}) > 0$.

(ii) If

$$\mathbf{P} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{2}{7} & \frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

then $\alpha_{\min} = \frac{\log 2}{\log 3}$ and $S_{\min} = \{11, 22\}$. Since $11 \cap 22 = \emptyset$ therefore $\mathcal{D}^{(\mu)}(\alpha_{\min}) = 0$.

The following well-known corollary is derived from Theorem 4.1 immediately.

Corollary 4.1. *Assume that all entries of* $\mathbf{A}$ *are equal to 1 and* μ *is the Bernoulli measure corresponding to the Bernoulli matrix* $\left(\begin{smallmatrix} b_1 & \dots & b_N \\ & \dots & \\ b_1 & \dots & b_N \end{smallmatrix}\right)$. *Then* $\mathcal{D}^{(\mu)}(\alpha_{\min}) = 0$ *if and only if* $\#\{1 \le i \le N \mid \frac{\log b_i}{\log r_i} = \alpha_{\min}\} = 1$. *Moreover if* $\{1 \le i \le N \mid \frac{\log b_i}{\log r_i} = \alpha_{\min}\} = \{i_0\}$ *then* $S_{\min} = \{i_0 i_0\}$.

Second corollary shows that "typical" spectra are nondegenerate. $\mathcal{M}_1$ denotes the space consisting of all Markov measures on K and M_1 denotes the set consisting of all $N \times N$ stochastic matrices $\mathbf{P}$ satisfying that $P(ij) > 0$ if and only if $A(ij) = 1$ for all $i, j = 1, ..., N$. $\mathcal{M}_1$ is equipped with the weak* topology and M_1 is equipped with the relative topology induced by the Euclid space $\mathbb{R}^{N^2}$. For $P \in M_1$, we denote by $\varphi(P)$ the Markov measure corresponding to P. The map $\varphi : M_1 \to \mathcal{M}_1$ is a homeomorphism. We set

$$G = \left\{ \mathbf{P} \in M_1 \mid \frac{\log P(w)}{\log r(w)} \neq \frac{\log P(w')}{\log r(w')} \text{ for any two } w, w' \in S \text{ with } w \not\sim_{\text{rot}} w' \right\}.$$

It is easy to see that G is an open and dense subset of M_1. Theorem 4.1 tells us that $\varphi(G) \subset \{\mu \in \mathcal{M}_1 \mid \mathcal{D}^{(\mu)}(\alpha_{\min}) = 0\}$, and hence, we obtain the following:

Corollary 4.2. $\{\mu \in \mathcal{M}_1 \mid \mathcal{D}^{(\mu)}(\alpha_{\min}) = 0\}$ *contains an open and dense subset of* $\mathcal{M}_1$.

Schmeling showed in [11] that the space of all Hölder continuous functions defined on a common mixing subshift with nondegenerate spectra contains a residual set.

5. Proof of Main Theorems

5.1. *Aim and setting*

The aim of this chapter is to prove Theorem 1.1 and 1.2. Let $\mathbf{A}$ be one of the three matrices in (1.2). We actually work on the topological Markov shift $\Sigma_{\mathbf{A}}^+$.

Let μ and $\widehat{\mu}$ be Markov measures on $\Sigma_{\mathbf{A}}^+$ corresponding to stochastic matrices $\mathbf{P} = (P(ij))$ and $\widehat{\mathbf{P}} = (\widehat{P}(ij))$, respectively. Fix $r_1, r_2, \widehat{r}_1, \widehat{r}_2 \in (0, 1)$. We define two functions $\beta, \widehat{\beta} : \mathbb{R} \to \mathbb{R}$ by

$$\text{both of } (P(ij)^q \, r_j^{-\beta(q)}),\ (\widehat{P}(ij)^q \, \widehat{r}_j^{-\widehat{\beta}(q)}) \text{ have spectral radius 1.}$$

Recall that

$$\lim_{q \to +\infty} \beta'(q) = \alpha_{\min}, \quad \lim_{q \to -\infty} \beta'(q) = \alpha_{\max},$$

where $\alpha_{\min}, \alpha_{\max}$ are defined by (2.5). In what follows, we always assume that

$$\beta = \widehat{\beta} \quad \text{and} \quad \alpha_{\min} < \alpha_{\max}.$$

We define the sets $\widehat{S}_{\min}$ and $\widehat{S}_{\max}$ for $\widehat{\mu}$ and $\widehat{r}_1, \widehat{r}_2$ as (4.2). We set the following condition:

(D)$'$. There exists $\rho \in \mathrm{Aut}(\Sigma_{\mathbf{A}}^+)$ such that

$$\widehat{\mu} = \mu \circ \rho \quad \text{and} \quad \widehat{r} = r \circ \rho,$$

where $r, \widehat{r} : \Sigma_{\mathbf{A}}^+ \to \mathbb{R}$ are defined by $r(\omega) = r_{\omega_1}, \widehat{r}(\omega) = \widehat{r}_{\omega_1}$.

In the next two sections, we prove the following two theorems. Theorem 1.1 and 1.2 immediately follow from Theorem 5.1 and 5.2, respectively. The exceptional measures in 3.3 appears again.

Theorem 5.1. *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *We have the following:*

(i) If $\beta \neq \log_r(\lambda^q + (1-\lambda)^q)$ *for any* $\lambda \in (0, 1/2)$ *and* $r \in (0,1)$ *then* (D)$'$ *holds.*

(ii) If $\beta = \log_r(\lambda^q + (1-\lambda)^q)$ *for some* $\lambda \in (0, 1/2)$ *and* $r \in (0,1)$ *then the following hold:*

(a) $r_1 = r_2 = r$ *and* $\mathbf{P}$ *coincides with one of the following four matrices:*

$$\begin{pmatrix} 1-\lambda & \lambda \\ 1-\lambda & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}, \begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix}.$$

(b) (D)$'$ *holds with* $\rho = \mathrm{id}$ *if and only if both* $\mathbf{P}$ *and* $\widehat{\mathbf{P}}$ *coincide with the same matrix in (a).* (D)$'$ *holds with* $\rho = \mathrm{flip}$ *if and only if either* $(\mathbf{P}, \widehat{\mathbf{P}})$ *or* $(\widehat{\mathbf{P}}, \mathbf{P})$ *coincides with* $\left(\left(\begin{smallmatrix} 1-\lambda & \lambda \\ 1-\lambda & \lambda \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{smallmatrix}\right)\right)$.

Theorem 5.2. *Assume that* $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *or* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. *Then* (D)$'$ *holds with* $\rho = \mathrm{id}$.

5.2. *Proof of Theorem 5.1*

Assume that $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Then $S = \{11, 22, 121, 212\}$. We omit writing 212 since $212 \sim_{\mathrm{rot}} 121$. $(S_{\min}, S_{\max})$ gives us a pair of two nonempty disjoint subsets of $\{11, 22, 121\}$. The total number of such pairs is twelve, and by Theorem 4.1, we can divide these pairs into the following three cases:

Case 1. $\underline{\beta^*(\alpha_{\min}) > 0 \text{ and } \beta^*(\alpha_{\max}) = 0}$. In this case, $(S_{\min}, S_{\max})$ coincides with one of the following two:

$$\text{(S1) } (\{11, 121\}, \{22\}), \quad \text{(S2) } (\{22, 121\}, \{11\}).$$

Case 2. $\underline{\beta^*(\alpha_{\min}) = 0 \text{ and } \beta^*(\alpha_{\max}) > 0}$. In this case, $(S_{\min}, S_{\max})$ coincides with one of the following two:

$$\text{(S3) } (\{11\}, \{22, 121\}), \quad \text{(S4) } (\{22\}, \{11, 121\}).$$

Case 3. $\underline{\beta^*(\alpha_{\min}) = \beta^*(\alpha_{\max}) = 0}$. In this case, $(S_{\min}, S_{\max})$ coincides with one of the following eight:

$$\begin{aligned}
&(\text{S5})\ (\{11\},\,\{22\}), \quad (\text{S6})\ (\{22\},\,\{11\}),\\
&(\text{S7})\ (\{11\},\,\{121\}), \quad (\text{S8})\ (\{22\},\,\{121\}),\\
&(\text{S9})\ (\{121\},\,\{11\}), \quad (\text{S10})\ (\{121\},\,\{22\}),\\
&(\text{S11})\ (\{121\},\,\{11,22\}), \quad (\text{S12})\ (\{11,22\},\,\{121\}).
\end{aligned}$$

For each $\rho \in \mathrm{Aut}(\Sigma_2^+)$, we define $\tilde{\rho} : S \to S$ by

$$\text{if } \rho = \mathrm{id} \text{ then } \tilde{\rho} : \begin{cases} 11 \mapsto 11, \\ 22 \mapsto 22, \\ 121 \mapsto 121, \\ 212 \mapsto 212, \end{cases} \qquad \text{if } \rho = \mathrm{flip} \text{ then } \tilde{\rho} : \begin{cases} 11 \mapsto 22, \\ 22 \mapsto 11, \\ 121 \mapsto 212, \\ 212 \mapsto 121. \end{cases}$$

We define the equivalence relation $\sim_{\mathrm{Aut}}$ on $\{(\text{S1}), ..., (\text{S12})\}$ by

$$\begin{aligned}
&(S_1,\, S_2) \sim_{\mathrm{Aut}} (S_1',\, S_2')\\
&\iff \text{there exists } \rho \in \mathrm{Aut}(\Sigma_2^+) \text{ such that } (S_1,\, S_2) = (\tilde{\rho}(S_1'),\, \tilde{\rho}(S_2')).
\end{aligned}$$

(S1), (S3), (S5), (S7), (S9), (S11), (S12) are not equivalent each other and we have

$$\begin{aligned}
&(\text{S1}) \sim_{\mathrm{Aut}} (\text{S2}), \quad (\text{S3}) \sim_{\mathrm{Aut}} (\text{S4}), \quad (\text{S5}) \sim_{\mathrm{Aut}} (\text{S6}),\\
&(\text{S7}) \sim_{\mathrm{Aut}} (\text{S8}), \quad (\text{S9}) \sim_{\mathrm{Aut}} (\text{S10}).
\end{aligned}$$

We tabulate the values of $\alpha_{\min}$ and $\alpha_{\max}$ of each representative element. See Table 1.

Table 1. The values of $\alpha_{\min}$ and $\alpha_{\max}$

	$\alpha_{\min}$	$\alpha_{\max}$
(S1)	$\frac{\log P(11)}{\log r_1},\ \frac{\log P(121)}{\log r_1 r_2}$	$\frac{\log P(22)}{\log r_2}$
(S3)	$\frac{\log P(11)}{\log r_1}$	$\frac{\log P(22)}{\log r_2},\ \frac{\log P(121)}{\log r_1 r_2}$
(S5)	$\frac{\log P(11)}{\log r_1}$	$\frac{\log P(22)}{\log r_2}$
(S7)	$\frac{\log P(11)}{\log r_1}$	$\frac{\log P(121)}{\log r_1 r_2}$
(S9)	$\frac{\log P(121)}{\log r_1 r_2}$	$\frac{\log P(11)}{\log r_1}$
(S11)	$\frac{\log P(121)}{\log r_1 r_2}$	$\frac{\log P(11)}{\log r_1},\ \frac{\log P(22)}{\log r_2}$
(S12)	$\frac{\log P(11)}{\log r_1},\ \frac{\log P(22)}{\log r_2}$	$\frac{\log P(121)}{\log r_1 r_2}$

Since the matrices $(P(ij)^q\, r_j^{-\beta(q)})$ and $(\widehat{P}(ij)^q\, \widehat{r}_j^{\,-\beta(q)})$ have eigenvalue 1, we have

$$(r_1 r_2)^{q\frac{\log P(121)}{\log r_1 r_2} - \beta(q)} = \left(1 - r_1^{\,q\frac{\log P(11)}{\log r_1} - \beta(q)}\right)\left(1 - r_2^{\,q\frac{\log P(22)}{\log r_2} - \beta(q)}\right) \tag{5.1}$$

and

$$(\widehat{r}_1 \widehat{r}_2)^{q\frac{\log \widehat{P}(121)}{\log \widehat{r}_1 \widehat{r}_2} - \beta(q)} = \left(1 - \widehat{r}_1^{\,q\frac{\log \widehat{P}(11)}{\log \widehat{r}_1} - \beta(q)}\right)\left(1 - \widehat{r}_2^{\,q\frac{\log \widehat{P}(22)}{\log \widehat{r}_2} - \beta(q)}\right) \tag{5.2}$$

for each $q \in \mathbb{R}$. In particular, we have Moran's formula

$$r_1^s + r_2^s = \widehat{r}_1^s + \widehat{r}_2^s = 1,$$

where $s = -\beta(0)$.

Lemma 5.1. *Assume that $\beta^*(\alpha_{\min}) > 0$ or $\beta^*(\alpha_{\max}) > 0$. Both $\mathbf{P}$ and (r_1, r_2) are uniquely determined as follows:*

(i) If $(S_{\min}, S_{\max}) = (\mathrm{S1})$ then

$$(r_1, r_2) = (\gamma^{a/s},\ (1-\gamma^a)^{1/s}), \quad \mathbf{P} = \begin{pmatrix} r_1^{\alpha_{\min}} & 1 - r_1^{\alpha_{\min}} \\ 1 - r_2^{\alpha_{\max}} & r_2^{\alpha_{\max}} \end{pmatrix} \tag{5.3}$$

and if $(S_{\min}, S_{\max}) = (\mathrm{S2})$ then

$$(r_1, r_2) = ((1-\gamma^a)^{1/s},\ \gamma^{a/s}), \quad \mathbf{P} = \begin{pmatrix} r_1^{\alpha_{\max}} & 1 - r_1^{\alpha_{\max}} \\ 1 - r_2^{\alpha_{\min}} & r_2^{\alpha_{\min}} \end{pmatrix}, \tag{5.4}$$

where we put $a = s/\beta^*(\alpha_{\min})$ and γ is a unique real number such that $0 < \gamma < 1$ and $\gamma^a + \gamma^{-a}(1-\gamma)^a = 1$.

(ii) If $(S_{\min}, S_{\max}) = (\mathrm{S3})$ or $(\mathrm{S4})$ then we have (5.3) or (5.4), respectively, by putting $a = s/\beta^*(\alpha_{\max})$.

Proof. (i). Assume that $(S_{\min}, S_{\max}) = (\mathrm{S1})$. Table 1 and (5.1) tell us that

$$(r_1 r_2)^{q\alpha_{\min} - \beta(q)} = (1 - r_1^{q\alpha_{\min} - \beta(q)})(1 - r_2^{q\alpha_{\max} - \beta(q)})$$

holds for each $q \in \mathbb{R}$. By letting $q \to +\infty$ in this equation, we obtain

$$(r_1 r_2)^{\beta^*(\alpha_{\min})} = 1 - r_1^{\beta^*(\alpha_{\min})}.$$

By combining this equation with Moran's formula, we obtain

$$\left(r_1^{\beta^*(\alpha_{\min})}\right)^a + \left(\frac{1}{r_1^{\beta^*(\alpha_{\min})}} - 1\right)^a = 1.$$

We can easily check that $a > 1$. The equation $x^{-a} + (x-1)^a = 1$ has a unique solution $x > 1$ for each $a > 1$. Hence, we obtain (5.3). By changing the roles of r_1 and r_2, we obtain (5.4).

(ii). We can use the same argument as (i) just by letting $q \to -\infty$ instead of letting $q \to +\infty$. $\square$

Lemma 5.2. *If $(S_{\min},\, S_{\max}) = (\widehat{S}_{\min},\, \widehat{S}_{\max})$ and $(S_{\min},\, S_{\max}) = (\mathrm{S5})$, $(\mathrm{S7})$, $(\mathrm{S9})$, $(\mathrm{S11})$ or $(\mathrm{S12})$ then $(\mathrm{D})'$ holds.*

Proof. Assume that $(S_{\min},\, S_{\max}) = (\mathrm{S5})$. Put $\alpha = \frac{\log P(121)}{\log r_1 r_2}$ and $\widehat{\alpha} = \frac{\log \widehat{P}(121)}{\log \widehat{r}_1 \widehat{r}_2}$. By Table 1, (5.1) and (5.2), we have

$$\frac{(1 - r_1^{q\alpha_{\min} - \beta(q)})(1 - r_2^{q\alpha_{\max} - \beta(q)})}{(1 - \widehat{r}_1^{q\alpha_{\min} - \beta(q)})(1 - \widehat{r}_2^{q\alpha_{\max} - \beta(q)})} = \frac{(r_1 r_2)^{q\alpha - \beta(q)}}{(\widehat{r}_1 \widehat{r}_2)^{q\widehat{\alpha} - \beta(q)}}$$

for each $q \in \mathbb{R}$. Therefore, by Lemma 4.3, we have

$$\frac{\log r_1}{\log \widehat{r}_1} = \lim_{q \to +\infty} \frac{1 - r_1^{q\alpha_{\min} - \beta(q)}}{1 - \widehat{r}_1^{q\alpha_{\min} - \beta(q)}} = \lim_{q \to +\infty} \left(\frac{(r_1 r_2)^{\alpha - \alpha_{\min}}}{(\widehat{r}_1 \widehat{r}_2)^{\widehat{\alpha} - \alpha_{\min}}} \right)^q.$$

Since $\lim_{q \to +\infty} \left(\frac{(r_1 r_2)^{\alpha - \alpha_{\min}}}{(\widehat{r}_1 \widehat{r}_2)^{\widehat{\alpha} - \alpha_{\min}}} \right)^q$ is equal to 0 or 1 or $+\infty$, we have $\frac{\log r_1}{\log \widehat{r}_1} = 1$, that is, $r_1 = \widehat{r}_1$. Once we obtain $r_1 = \widehat{r}_1$, we show by Moran's formula and Table 1 that $(r_1, r_2) = (\widehat{r}_1, \widehat{r}_2)$ and $\mathbf{P} = \widehat{\mathbf{P}}$. We conclude that $(\mathrm{D})'$ holds with $\rho = \mathrm{id}$.

Assume that $(S_{\min}, S_{\max}) = $ (S7). Put $\alpha = \frac{\log P(22)}{\log r_2}$ and $\widehat{\alpha} = \frac{\log \widehat{P}(22)}{\log \widehat{r}_2}$. By Table 1, (5.1) and (5.2), we have

$$\frac{(1 - r_1^{q\alpha_{\min} - \beta(q)})(1 - r_2^{q\alpha - \beta(q)})}{(1 - \widehat{r}_1^{q\alpha_{\min} - \beta(q)})(1 - \widehat{r}_2^{q\widehat{\alpha} - \beta(q)})} = \frac{(r_1 r_2)^{q\alpha_{\max} - \beta(q)}}{(\widehat{r}_1 \widehat{r}_2)^{q\alpha_{\max} - \beta(q)}}$$

for each $q \in \mathbb{R}$. Therefore, by Lemma 4.3, we have

$$\frac{\log r_1}{\log \widehat{r}_1} = \lim_{q \to +\infty} \frac{1 - r_1^{q\alpha_{\min} - \beta(q)}}{1 - \widehat{r}_1^{q\alpha_{\min} - \beta(q)}} = \lim_{q \to +\infty} \left(\left(\frac{r_1 r_2}{\widehat{r}_1 \widehat{r}_2} \right)^{\alpha_{\max} - \alpha_{\min}} \right)^q,$$

and thus, we have $\frac{\log r_1}{\log \widehat{r}_1} = 1$. Therefore we have $(r_1, r_2) = (\widehat{r}_1, \widehat{r}_2)$ and $\mathbf{P} = \widehat{\mathbf{P}}$, and we conclude that $(\mathrm{D})'$ holds with $\rho = \mathrm{id}$. If $(S_{\min}, S_{\max}) = $ (S9), then we can use a similar argument by letting $q \to -\infty$ instead of letting $q \to +\infty$, and we conclude that $(\mathrm{D})'$ holds with $\rho = \mathrm{id}$.

Assume that $(S_{\min}, S_{\max}) = $ (S11). By Table 1, (5.1) and (5.2), we have

$$\frac{(1 - r_1^{q\alpha_{\max} - \beta(q)})(1 - r_2^{q\alpha_{\max} - \beta(q)})}{(1 - \widehat{r}_1^{q\alpha_{\max} - \beta(q)})(1 - \widehat{r}_2^{q\alpha_{\max} - \beta(q)})} = \frac{(r_1 r_2)^{q\alpha_{\min} - \beta(q)}}{(\widehat{r}_1 \widehat{r}_2)^{q\alpha_{\min} - \beta(q)}}$$

for each $q \in \mathbb{R}$. By applying Lemma 4.3 to this equation with letting $q \to -\infty$, we obtain $\frac{r_1 r_2}{\widehat{r}_1 \widehat{r}_2} = 1$. Thus, by Moran's formula, we have

$$r_1^s(1 - r_1^s) = (r_1 r_2)^s = (\widehat{r}_1 \widehat{r}_2)^s = \widehat{r}_1^s(1 - \widehat{r}_1^s).$$

This shows that both r_1^s and $\widehat{r}_1^s$ are the roots of the same quadratic equation, and hence, $r_1 = \widehat{r}_1$ or $r_1 = \widehat{r}_2$. In the former case, $(\mathrm{D})'$ holds with $\rho = \mathrm{id}$. In the latter case, $(\mathrm{D})'$ holds with $\rho = \mathrm{flip}$. We can treat the case where $(S_{\min}, S_{\max}) = $ (S12) in a similar manner, by letting $q \to +\infty$ instead of letting $q \to -\infty$. $\square$

Lemma 5.3. *Let $a_1, ..., a_m > 0$. We write $a = \max_{1 \le i \le m} a_i$ and $M = \#\{1 \le i \le m \mid a_i = a\}$. Then the following hold:*

(i) If functions $\phi_1, ..., \phi_m : \mathbb{R} \to \mathbb{R}$ satisfy $\phi_1(q) \sim a_1^q, ..., \phi_m(q) \sim a_m^q$ $(q \to +\infty)$ then $\phi_1 + \cdots + \phi_m \sim a_1^q + \cdots + a_m^q \sim M \times a^q$ $(q \to +\infty)$.

(ii) Let $b_1, ..., b_n > 0$ and we write $b = \max_{1 \le i \le n} b_i$, $N = \#\{1 \le i \le n \mid b_i = b\}$. When $q \to +\infty$, we have

$$\frac{a_1^q + \cdots + a_m^q}{b_1^q + \cdots + b_n^q} \to \begin{cases} 0 & (a < b), \\ +\infty & (a > b), \\ \frac{M}{N} & (a = b). \end{cases}$$

Proof. Easy. $\square$

Lemma 5.4. *If* $(S_{\min}, S_{\max}) = $ (S7) *then* $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S12).

Proof. Put $\alpha = \frac{\log P(22)}{\log r_2}$. Assume that $(\widehat{S}_{\min}, \widehat{S}_{\max}) = $ (S12). We observe by Table 1, (5.1) and (5.2) that

$$\left(\frac{r_1 r_2}{\sqrt{\widehat{r}_1 \widehat{r}_2}}\right)^{q\alpha_{\max}-\beta(q)} = \sqrt{\frac{1 - r_1^{q\alpha_{\min}-\beta(q)}}{1 - \widehat{r}_1^{q\alpha_{\min}-\beta(q)}}}\sqrt{\frac{1 - r_1^{q\alpha_{\min}-\beta(q)}}{1 - \widehat{r}_2^{q\alpha_{\min}-\beta(q)}}}\,(1 - r_2^{q\alpha-\beta(q)}) \tag{5.5}$$

and

$$\frac{1 - (r_1 r_2)^{q\alpha_{\max}-\beta(q)}}{1 - (\widehat{r}_1 \widehat{r}_2)^{q\alpha_{\max}-\beta(q)}} = \frac{r_1^{q\alpha_{\min}-\beta(q)} + r_2^{q\alpha-\beta(q)}(1 - r_1^{q\alpha_{\min}-\beta(q)})}{\widehat{r}_1^{q\alpha_{\min}-\beta(q)} + \widehat{r}_2^{q\alpha_{\min}-\beta(q)}(1 - \widehat{r}_1^{q\alpha_{\min}-\beta(q)})} \tag{5.6}$$

for each $q \in \mathbb{R}$.

By applying Lemma 4.3 to (5.5) with $q \to +\infty$, we have

$$\frac{r_1 r_2}{\sqrt{\widehat{r}_1 \widehat{r}_2}} = \sqrt{\frac{\log r_1}{\log \widehat{r}_1}}\sqrt{\frac{\log r_1}{\log \widehat{r}_2}} = 1. \tag{5.7}$$

(5.7) shows that $\frac{\log r_1 r_2}{\log \widehat{r}_1 \widehat{r}_2} = \frac{1}{2}$. Therefore, by Lemma 4.3, we have

$$\begin{aligned}
\frac{1}{2} &= \lim_{q\to-\infty} \frac{r_1^{q\alpha_{\min}-\beta(q)} + r_2^{q\alpha-\beta(q)}(1 - r_1^{q\alpha_{\min}-\beta(q)})}{\widehat{r}_1^{q\alpha_{\min}-\beta(q)} + \widehat{r}_2^{q\alpha_{\min}-\beta(q)}(1 - \widehat{r}_1^{q\alpha_{\min}-\beta(q)})} \\
&= \lim_{q\to-\infty} \frac{(r_1^{\alpha_{\min}-\alpha_{\max}})^q + (r_2^{\alpha-\alpha_{\max}})^q}{(\widehat{r}_1^{\alpha_{\min}-\alpha_{\max}})^q + (\widehat{r}_2^{\alpha_{\min}-\alpha_{\max}})^q}.
\end{aligned}$$

Thus, by Lemma 5.3 (ii), we have

$$\widehat{r}_1^{\alpha_{\min}-\alpha_{\max}} = \widehat{r}_2^{\alpha_{\min}-\alpha_{\max}}, \quad \text{that is,} \quad \widehat{r}_1 = \widehat{r}_2.$$

By combining $\widehat{r}_1 = \widehat{r}_2$ with (5.7), we obtain the contradiction $r_2 = 1$, and we conclude that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S12). $\square$

Lemma 5.5. *If* $(S_{\min}, S_{\max}) = $ (S7) *then* $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S9) *and* (S11).

Proof. Assume that $(\widehat{S}_{\min}, \widehat{S}_{\max}) = $ (S9) or (S11). Put $\alpha = \frac{\log P(22)}{\log r_2}$ and $\widehat{\alpha} = \frac{\log \widehat{P}(22)}{\log \widehat{r}_2}$. By Table 1, (5.1) and (5.2), we have

$$1 - (\widehat{r}_1 \widehat{r}_2)^{q\alpha_{\min}-\beta(q)} = \widehat{r}_1^{q\alpha_{\max}-\beta(q)}(1 - \widehat{r}_2^{q\widehat{\alpha}-\beta(q)}) + \widehat{r}_2^{q\widehat{\alpha}-\beta(q)} \tag{5.8}$$

for each $q \in \mathbb{R}$ and

$$\alpha_{\min} < \widehat{\alpha}. \tag{5.9}$$

By (5.8), we have

$$\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\min}-\beta(q)}}{1-r_1^{q\alpha_{\min}-\beta(q)}} = \left(\frac{\widehat{r}_1^{q\alpha_{\max}-\beta(q)}(1-\widehat{r}_2^{q\widehat{\alpha}-\beta(q)})}{(r_1r_2)^{q\alpha_{\max}-\beta(q)}} + \frac{\widehat{r}_2^{q\widehat{\alpha}-\beta(q)}}{(r_1r_2)^{q\alpha_{\max}-\beta(q)}}\right) \times (1-r_2^{q\alpha-\beta(q)})$$

for each $q \in \mathbb{R}$. We observe by (5.9) that both $q\alpha-\beta(q)$ and $q\widehat{\alpha}-\beta(q)$ tend to $+\infty$ when $q \to +\infty$, therefore, both $1-r_2^{q\alpha-\beta(q)}$ and $1-r_2^{q\widehat{\alpha}-\beta(q)}$ are lager than $1/2$ for sufficiently large $q>0$. Moreover $0 < q\widehat{\alpha}-\beta(q) \le q\alpha_{\max}-\beta(q)$ for each $q>0$. Thus, for sufficiently large $q>0$, we have

$$\begin{aligned}\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\min}-\beta(q)}}{1-r_1^{q\alpha_{\min}-\beta(q)}} &\ge \frac{1}{2}\max\left(\frac{\widehat{r}_1^{q\alpha_{\max}-\beta(q)}(1-\widehat{r}_2^{q\widehat{\alpha}-\beta(q)})}{(r_1r_2)^{q\alpha_{\max}-\beta(q)}},\ \frac{\widehat{r}_2^{q\widehat{\alpha}-\beta(q)}}{(r_1r_2)^{q\alpha_{\max}-\beta(q)}}\right)\\ &\ge \frac{1}{4}\max\left(\frac{\widehat{r}_1^{q\alpha_{\max}-\beta(q)}}{(r_1r_2)^{q\alpha_{\max}-\beta(q)}},\ \frac{\widehat{r}_2^{q\alpha_{\max}-\beta(q)}}{(r_1r_2)^{q\alpha_{\max}-\beta(q)}}\right)\\ &= \frac{1}{4}\max\left(\left(\frac{\widehat{r}_1}{r_1r_2}\right)^{q\alpha_{\max}-\beta(q)},\ \left(\frac{\widehat{r}_2}{r_1r_2}\right)^{q\alpha_{\max}-\beta(q)}\right).\end{aligned}$$

If $r_1 \le \widehat{r}_1$, then $\frac{\widehat{r}_1}{r_1r_2} \ge \frac{1}{r_2} > 1$, and thus, $\lim_{q\to+\infty}\left(\frac{\widehat{r}_1}{r_1r_2}\right)^{q\alpha_{\max}-\beta(q)} = +\infty$. If $r_1 > \widehat{r}_1$, then $r_2^s = 1-r_1^s < 1-\widehat{r}_1^s = \widehat{r}_2^s$, and hence, we have $\frac{\widehat{r}_2}{r_1r_2} > \frac{1}{r_1} > 1$ which implies $\lim_{q\to+\infty}\left(\frac{\widehat{r}_2}{r_1r_2}\right)^{q\alpha_{\max}-\beta(q)} = +\infty$. Therefore, $\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\min}-\beta(q)}}{1-r_1^{q\alpha_{\min}-\beta(q)}}$ tends to $+\infty$ when $q \to +\infty$. However, by Lemma 4.3, we have $\lim_{q\to+\infty}\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\min}-\beta(q)}}{1-r_1^{q\alpha_{\min}-\beta(q)}} = \frac{\log\widehat{r}_1\widehat{r}_2}{\log r_1} < +\infty$. We obtain a contradiction, and hence, we conclude that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S9) and (S11). $\square$

Lemma 5.6. *If* $(S_{\min}, S_{\max}) =$ (S7) *then* $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S5).

Proof. Assume that $(\widehat{S}_{\min}, \widehat{S}_{\max}) =$ (S5). Put $\alpha = \frac{\log P(22)}{\log r_2}$ and $\widehat{\alpha} = \frac{\log \widehat{P}(121)}{\log \widehat{r}_1\widehat{r}_2}$. We observe by Table 1, (5.1) and (5.2) that

$$\frac{1-r_1^{q\alpha_{\min}-\beta(q)}}{1-\widehat{r}_1^{q\alpha_{\min}-\beta(q)}}\cdot\frac{1-r_2^{q\alpha-\beta(q)}}{1-\widehat{r}_2^{q\alpha_{\max}-\beta(q)}} = \frac{(r_1r_2)^{q\alpha_{\max}-\beta(q)}}{(\widehat{r}_1\widehat{r}_2)^{q\widehat{\alpha}-\beta(q)}}$$

for each $q \in \mathbb{R}$. Therefore, by Lemma 4.3, we have

$$\frac{\log r_1}{\log\widehat{r}_1} = \lim_{q\to+\infty}\frac{1-r_1^{q\alpha_{\min}-\beta(q)}}{1-\widehat{r}_1^{q\alpha_{\min}-\beta(q)}} = \lim_{q\to+\infty}\left(\frac{(r_1r_2)^{\alpha_{\max}-\alpha_{\min}}}{(\widehat{r}_1\widehat{r}_2)^{\widehat{\alpha}-\alpha_{\min}}}\right)^q,$$

and thus, we obtain

$$\frac{(r_1r_2)^{\alpha_{\max}-\alpha_{\min}}}{(\widehat{r}_1\widehat{r}_2)^{\widehat{\alpha}-\alpha_{\min}}} = \frac{\log r_1}{\log\widehat{r}_1} = 1.$$

This and Moran's formula imply that $r_1 = \widehat{r}_1$ and $r_2 = \widehat{r}_2$. Therefore, we have

$$(r_1r_2)^{\alpha_{\max}-\widehat{\alpha}} = \frac{(r_1r_2)^{\alpha_{\max}-\alpha_{\min}}}{(\widehat{r}_1\widehat{r}_2)^{\widehat{\alpha}-\alpha_{\min}}} = 1,$$

which contradicts $\alpha_{\max} - \widehat{\alpha} > 0$. We conclude that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S5). $\square$

Lemma 5.7. *If* $(S_{\min}, S_{\max}) =$ (S9) *then* $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S5), (S7), (S11) *and* (S12).

Proof. We can use the same arguments as that in the proofs of Lemma 5.4–5.6, by letting $q \to -\infty$ instead of letting $q \to +\infty$. $\square$

Lemma 5.8. *Assume that* $\widehat{r}_1 \neq \widehat{r}_2$. *If* $(S_{\min}, S_{\max}) =$ (S5) *then* $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S11) *and* (S12).

Proof. Put $\alpha = \frac{\log P(121)}{\log r_1r_2}$.

Assume that $(\widehat{S}_{\min}, \widehat{S}_{\max}) =$ (S11). By Table 1, (5.1) and (5.2), we have

$$\frac{(r_1r_2)^{q\alpha-\beta(q)}}{(\sqrt{\widehat{r}_1\widehat{r}_2})^{q\alpha_{\min}-\beta(q)}} = \sqrt{\frac{1-r_2^{q\alpha_{\max}-\beta(q)}}{1-\widehat{r}_1^{q\alpha_{\max}-\beta(q)}}}\sqrt{\frac{1-r_2^{q\alpha_{\max}-\beta(q)}}{1-\widehat{r}_2^{q\alpha_{\max}-\beta(q)}}}\,(1-r_1^{q\alpha_{\min}-\beta(q)}) \tag{5.10}$$

and

$$\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\min}-\beta(q)}}{1-r_1^{q\alpha_{\min}-\beta(q)}} = \frac{\widehat{r}_1^{q\alpha_{\max}-\beta(q)} + \widehat{r}_2^{q\alpha_{\max}-\beta(q)} - (\widehat{r}_1\widehat{r}_2)^{q\alpha_{\max}-\beta(q)}}{(r_1r_2)^{q\alpha-\beta(q)}/(1-r_2^{q\alpha_{\max}-\beta(q)})} \tag{5.11}$$

for each $q \in \mathbb{R}$.

By letting $q \to -\infty$ in (5.10) we obtain

$$\frac{(r_1r_2)^{\alpha-\alpha_{\max}}}{(\sqrt{\widehat{r}_1\widehat{r}_2})^{\alpha_{\min}-\alpha_{\max}}} = \sqrt{\frac{\log r_2}{\log \widehat{r}_1}}\sqrt{\frac{\log r_2}{\log \widehat{r}_2}} = 1. \tag{5.12}$$

Assume that $\widehat{r}_1 < \widehat{r}_2$. We observe that

$$\text{the right hand side of (5.11)} \sim \left(\frac{\widehat{r}_2^{\alpha_{\max}-\alpha_{\min}}}{(r_1r_2)^{\alpha-\alpha_{\min}}}\right)^q$$

when $q \to +\infty$, and thus, we obtain

$$\frac{\log \widehat{r}_1\widehat{r}_2}{\log r_1} = \frac{\widehat{r}_2^{\alpha_{\max}-\alpha_{\min}}}{(r_1r_2)^{\alpha-\alpha_{\min}}} = 1. \tag{5.13}$$

We have $\widehat{r}_2^{\alpha_{\max}-\alpha_{\min}} = (r_1r_2)^{\alpha-\alpha_{\min}}$ by (5.13). Thus, by (5.12), we have $(r_1r_2)^{\alpha_{\max}-\alpha_{\min}} = (r_1r_2)^{\alpha_{\max}-\alpha} \times (r_1r_2)^{\alpha-\alpha_{\min}} = (\widehat{r}_2\sqrt{\widehat{r}_1\widehat{r}_2})^{\alpha_{\max}-\alpha_{\min}}$, that is,

$$r_1r_2 = \widehat{r}_2\sqrt{\widehat{r}_1\widehat{r}_2}.$$

This equation and (5.13) give us the contradiction $r_2 = \sqrt{\widehat{r}_2/\widehat{r}_1} > 1$, therefore, we conclude that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S11). We can show that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S12) by a similar argument. □

Lemma 5.9. *Assume that $\widehat{r}_1 \neq \widehat{r}_2$. The following hold:*
(i) If $(S_{\min}, S_{\max}) =$ (S11) then $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S5) and (S12).
(ii) If $(S_{\min}, S_{\max}) =$ (S12) then $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S5) and (S11).

Proof. We only discuss (i). Assume that $(\widehat{S}_{\min}, \widehat{S}_{\max}) =$ (S5). We must have $r_1 = r_2$ by Lemma 5.8.

Put $\widehat{\alpha} = \frac{\log \widehat{P}(121)}{\log \widehat{r}_1\widehat{r}_2}$. By Table 1, (5.1) and (5.2), we have

$$\frac{(\widehat{r}_1\widehat{r}_2)^{q\widehat{\alpha}-\beta(q)}}{(\sqrt{r_1r_2})^{q\alpha_{\min}-\beta(q)}} = \sqrt{\frac{1-\widehat{r}_2^{q\alpha_{\max}-\beta(q)}}{1-r_1^{q\alpha_{\max}-\beta(q)}}}\sqrt{\frac{1-\widehat{r}_2^{q\alpha_{\max}-\beta(q)}}{1-r_2^{q\alpha_{\max}-\beta(q)}}}\,(1-\widehat{r}_1^{q\alpha_{\min}-\beta(q)})$$

for each $q \in \mathbb{R}$, and hence, by letting $q \to -\infty$, we obtain $\sqrt{\frac{\log \widehat{r}_2}{\log r_1}}\sqrt{\frac{\log \widehat{r}_2}{\log r_2}} = 1$. This and $r_1 = r_2$ lead us to the contradiction $r_1 = r_2 = \widehat{r}_1 = \widehat{r}_2$, therefore we conclude that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S5).

Assume that $(\widehat{S}_{\min}, \widehat{S}_{\max}) =$ (S12). By Table 1, (5.1) and (5.2), we have

$$\begin{aligned}&\sqrt{\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\max}-\beta(q)}}{1-r_1^{q\alpha_{\max}-\beta(q)}}}\sqrt{\frac{1-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\max}-\beta(q)}}{1-r_2^{q\alpha_{\max}-\beta(q)}}}\\ =&\frac{\widehat{r}_1^{q\alpha_{\min}-\beta(q)}+\widehat{r}_2^{q\alpha_{\min}-\beta(q)}-(\widehat{r}_1\widehat{r}_2)^{q\alpha_{\min}-\beta(q)}}{(\sqrt{r_1r_2})^{q\alpha_{\min}-\beta(q)}}\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}&\sqrt{\frac{1-(r_1r_2)^{q\alpha_{\min}-\beta(q)}}{1-\widehat{r}_1^{q\alpha_{\min}-\beta(q)}}}\sqrt{\frac{1-(r_1r_2)^{q\alpha_{\min}-\beta(q)}}{1-\widehat{r}_2^{q\alpha_{\min}-\beta(q)}}}\\ =&\frac{r_1^{q\alpha_{\max}-\beta(q)}+r_2^{q\alpha_{\max}-\beta(q)}-(r_1r_2)^{q\alpha_{\max}-\beta(q)}}{(\sqrt{\widehat{r}_1\widehat{r}_2})^{q\alpha_{\max}-\beta(q)}}\end{aligned} \tag{5.15}$$

for each $q \in \mathbb{R}$.

Assume that $\widehat{r}_1 < \widehat{r}_2$. Then we have

$$\text{the right hand side of (5.14)} \sim \left(\left(\frac{\widehat{r}_2}{\sqrt{r_1r_2}}\right)^{\alpha_{\min}-\alpha_{\max}}\right)^q$$

when $q \to -\infty$, and thus, we obtain

$$\sqrt{\frac{\log \widehat{r}_1\widehat{r}_2}{\log r_1}}\sqrt{\frac{\log \widehat{r}_1\widehat{r}_2}{\log r_2}} = \frac{\widehat{r}_2}{\sqrt{r_1r_2}} = 1. \tag{5.16}$$

(5.16) implies that $(\log \widehat{r}_1\widehat{r}_2)^2 = \log r_1 \log r_2$ and $\widehat{r}_2 = \sqrt{r_1r_2}$. Thus, if $r_1 = r_2$, then we obtain the contradiction $r_1 = \widehat{r}_1\widehat{r}_2 < \widehat{r}_2 = r_1$. Moreover if $r_1r_2 \geq \widehat{r}_1\widehat{r}_2$, then $r_1, r_2 > \widehat{r}_1\widehat{r}_2$, and hence, we have $\frac{\log \widehat{r}_1\widehat{r}_2}{\log r_1}, \frac{\log \widehat{r}_1\widehat{r}_2}{\log r_2} > 1$ which contradicts (5.16). Therefore we have $r_1 \neq r_2$ and $r_1 r_2 < \widehat{r}_1\widehat{r}_2$.

Assume that $r_1 < r_2$. Then we have

$$\text{the right hand side of (5.15)} \sim \left(\left(\frac{r_2}{\sqrt{\widehat{r}_1\widehat{r}_2}}\right)^{\alpha_{\max}-\alpha_{\min}}\right)^q$$

when $q \to +\infty$, and hence, we obtain $\sqrt{\frac{\log r_1 r_2}{\log \widehat{r}_1}}\sqrt{\frac{\log r_1 r_2}{\log \widehat{r}_2}} = 1$. However, $r_1 r_2 < \widehat{r}_1\widehat{r}_2$ implies $\frac{\log r_1 r_2}{\log \widehat{r}_1}, \frac{\log r_1 r_2}{\log \widehat{r}_2} > 1$, which contradicts $\sqrt{\frac{\log r_1 r_2}{\log \widehat{r}_1}}\sqrt{\frac{\log r_1 r_2}{\log \widehat{r}_2}} = 1$. Therefore we conclude that $(\widehat{S}_{\min}, \widehat{S}_{\max}) \neq$ (S12). □

Corollary 5.1. *We have the following:*

(i) $(\mathrm{D})'$ *holds if and only if* $(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}} (\widehat{S}_{\min}, \widehat{S}_{\max})$.

(ii) Each of the following three guarantees that $(\mathrm{D})'$ *holds:*

(a) $(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}}$ (S1), (S3), (S7) *or* (S9).

(b) $r_1 \neq r_2$ *or* $\widehat{r}_1 \neq \widehat{r}_2$.

Proof. (i). 'Only if' part is clear. We will prove 'if' part. We may assume that $(S_{\min}, S_{\max}) = (\widehat{S}_{\min}, \widehat{S}_{\max})$. Lemma 5.1 and 5.2 tell us that $(\mathrm{D})'$ holds.

(ii). Assume that (a). By Lemma 5.1, we may assume that $(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}}$ (S7) or (S9). We obtain $(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}} (\widehat{S}_{\min}, \widehat{S}_{\max})$ by Lemma 5.4–5.7. We conclude by (i) that $(\mathrm{D})'$ holds.

Assume that (b). We only consider the case that $\widehat{r}_1 \neq \widehat{r}_2$. By (a), we may assume that $(S_{\min}, S_{\max}) =$ (S5), (S11) or (S12) and so does $(\widehat{S}_{\min}, \widehat{S}_{\max})$. We obtain $(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}} (\widehat{S}_{\min}, \widehat{S}_{\max})$ by Lemma 5.8 and 5.9. We conclude by (i) that $(\mathrm{D})'$ holds. □

Lemma 5.10. *Assume that* $r_1 = r_2 = r$. *The following hold:*

(i) If $\beta = \log_r(\lambda^q + (1-\lambda)^q)$ *for some* $\lambda \in (0, 1/2)$ *then* $(S_{\min}, S_{\max})$ *coincides with one of* (S5), (S6), (S11), (S12)*. Table 2 shows the possibilities for* $\mathbf{P}$.

Table 2. The possibilities for $\mathbf{P}$

$(S_{\min}, S_{\max})$	(S5)	(S6)	(S11)	(S12)
$\mathbf{P}$	$\begin{pmatrix} 1-\lambda & \lambda \\ 1-\lambda & \lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}$	$\begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}$	$\begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix}$

(ii) If $(S_{\min}, S_{\max}) =$ (S11) *or* (S12) *then* $\beta = \log_r(\lambda^q + (1-\lambda)^q)$ *for some* $\lambda \in (0, 1/2)$.

Proof. (i) is just a paraphrase of Theorem 3.3 (ii).

(ii). By Table 1, we can write $\mathbf{P} = \left(\begin{smallmatrix} b & 1-b \\ 1-b & b \end{smallmatrix}\right)$ for some $b \in (0,1), b \neq 1/2$, and hence, we have $\beta(q) = \log_r(b^q + (1-b)^q)$. □

Proof of Theorem 5.1. Note that Moran's formula tells us that $r_1 = r_2 = \widehat{r}_1 = \widehat{r}_2$ holds if and only if both $r_1 = r_2$ and $\widehat{r}_1 = \widehat{r}_2$ hold.

(i). Thanks to Corollary 5.1 (ii), we may assume that

$$r_1 = r_2 = \widehat{r}_1 = \widehat{r}_2.$$

We see by Lemma 5.10 (ii) that $(S_{\min}, S_{\max}) \neq$ (S11) and (S12). Moreover, by Corollary 5.1 (ii), we may assume that $(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}}$ (S5). This argument works on $(\widehat{S}_{\min}, \widehat{S}_{\max})$ and we obtain

$$(S_{\min}, S_{\max}) \sim_{\mathrm{Aut}} \text{(S5)} \sim_{\mathrm{Aut}} (\widehat{S}_{\min}, \widehat{S}_{\max}).$$

(ii). We prove (a). Assume that $r_1 \neq r_2$. From Lemma 5.10, there exists a Markov measure $\widetilde{\mu}$ on Σ_2^+ such that

$$(\widetilde{P}(ij)^q \, r^{-\beta(q)}) \text{ has spectral radius } 1,$$

where $\widetilde{\mathbf{P}} = (\widetilde{P}(ij))$ is the stochastic matrix corresponding to $\widetilde{\mu}$.

Corollary 5.1 (ii) tells us that (D)$'$ holds for μ and $\widetilde{\mu}$. In particular we have $\{r_1, r_2\} = \{r\}$ which contradicts $r_1 \neq r_2$. Thus, we conclude that $r_1 = r_2$. We obtain $r_1 = r$ by $2r_1^s = 1$ and $s = -\beta(0) = -\log_r 2$. We obtain $\widehat{r}_1 = \widehat{r}_2 = r$ similarly. The possibilities for $\mathbf{P}$ follow from Lemma 5.10 (i) immediately.

(b) immediately follows from (a) and Corollary 5.1 (i). We complete the proof of Theorem 5.1. □

5.3. *Proof of Theorem 1.2*

We only consider the case where $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We have $(S_{\min}, S_{\max}) = (\{11\}, \{121\})$ or $(\{121\}, \{11\})$ since $S = \{11, 121\}$, and hence, β^* is nondegenerate. We obtain $(r_1, r_2) = (\widehat{r}_1, \widehat{r}_2)$ and $\mathbf{P} = \widehat{\mathbf{P}}$ by arguments similar to that in the proofs of Lemma 5.5 and 5.2. We conclude that (D)$'$ holds with $\rho = \mathrm{id}$.

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